REFERENCES

1. Liapunov, A. M., General Problem of the Stability of Motion. Collected Works, Vol. 2. Moscow-Leningrad, Izd. Akad. Nauk SSSR, 1956.
2. Arnol'd, V. I., Small denominators and the stability problems for motions in classical and celestial mechanics. Uspekhi Matem. Nauk, Vol. 18, № 6, 1963.
3. Moser, J., Lectures on Hamiltonian Systems. Providence, New Jersey, Mem. Amer. Math. Soc., № 81, 1968.
4. Markeev, A. P., Stability of a canonical system with two degrees of freedom in the presence of resonance. PMM Vol. 32, № 4, 1968.
5. Gantmakher, F. R., Matrix Theory. Moscow, "Nauka", 1966.
6. Roels, J. and Louterman, G., Normalisation des systèmes linéaires canoniques et application au problème restreint des trois corps. Celest. Mech., Vol. 3, № 1, 1970.
7. Birkhoff, G. D., Dynamical Systems. Providence, Amer. Math. Soc., 1966.
8. Markeev, A. P., On the problem of stability of equilibrium positions of Hamiltonian systems. PMM Vol. 34, № 6, 1970.
9. Chetaev, N. G., The Stability of Motion (English translation). Pergamon Press, Book № 09505, 1961.
10. Rumiantsev, V. V., On asymptotic stability and instability of motion with respect to a part of the variables. PMM Vol. 35, № 1, 1971.
11. Landau, L. D. and Lifshits, E. M., Mechanics. Moscow, "Nauka", 1973. (See also English translation: Addison-Wesley Publishing Co., 1960).
12. Liapunov, A. M., Investigation of one of the singular cases of the problem of stability of motion. Collected Works, Vol. 2, Moscow, Izd. Akad. Nauk SSSR, 1956.
13. Moser, J., New aspects in the theory of stability of Hamiltonian systems. Commun. Pure Appl. Math., Vol. 11, № 1, 1958.

Translated by N. H. C.

UDC 531.31:534

ON SYNCHRONIZATION OF DYNAMIC SYSTEMS

PMM Vol. 38, № 5, 1974, pp. 800-809

A. S. GURTOVNIK and Iu. I. NEIMARK
(Gor'kii)

(Received December 17, 1973)

We introduce the concepts of the degree and the order of synchronism on the basis of a mathematical model of the emergence of synchronization in the form of an asymptotically stable integral torus in the phase plane. We investigate the existence conditions for synchronisms in a dynamic system described by differential equations with rapidly rotating phases. As an application we examine synchronisms in a system of quasi-Hamiltonian objects. In recent years the phenomena of synchronization and resonance in dynamic systems have been subjected to intensive study, in particular, in connection with the question of the synchronization of satellites [1, 2] and of mechanical vibrators [3]. On the mathematical side the appearance of synchronization is closely connected with the theory of differential equations with rapidly rotating phases. Here in the first place we must mention the works specified in [4−9].

1. Definition of synchronism of a dynamic system. Necessary conditions for synchronism. We define and investigate the conditions for the rise of synchronisms of different degrees and orders in a dynamic system described by the following differential equation:

$$\beta^{\cdot} = \omega(x) + \varepsilon B(\beta, x, \varepsilon), \qquad x^{\cdot} = \varepsilon X(\beta, x) + \varepsilon^2 Y(\beta, x, \varepsilon) \tag{1.1}$$

Here β and x are, respectively, r- and s-dimensional vectors, ε is a small positive parameter. All the functions occurring in the equations are assumed to be differentiable with respect to ε, twice continuously differentiable with respect to the components x_1, $x_2, \ldots, x_s$ of vector x, sufficiently smooth, and 2π-periodic in the components β_1, $\beta_2, \ldots, \beta_r$ of vector β. We say that system (1.1) admits of a synchronism of degree m if for all sufficiently small positive ε there exists in it an asymptotically-stable smooth integral toroidal surface of dimension $r - m$ of the form

$$x = x^\circ + \varepsilon h(\alpha_1, \alpha_2, \ldots, \alpha_{r-m}, \varepsilon), \qquad \psi = \psi^\circ + \varepsilon g(\alpha_1, \alpha_2, \ldots, \alpha_{r-m}, \varepsilon) \tag{1.2}$$

Here x° and ψ° are constant vectors, h and g are 2π-periodic smooth functions of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_{r-m}$, and the m-dimensional vector ψ is related to vector β by an integer $(m \times r)$-matrix P of rank m, so that

$$\psi = P\beta \tag{1.3}$$

Here, without loss of generality, we can assume that we take the components $\beta_{m+1}, \ldots, \beta_r$ of the vector β as the parameters $\alpha_1, \alpha_2, \ldots, \alpha_{r-m}$ and that

$$\Delta = \begin{vmatrix} p_{11}, & p_{12}, & \ldots, & p_{1m} \\ p_{m1}, & p_{m2}, & \ldots, & p_{mm} \end{vmatrix} \neq 0$$

Substituting (1.2), (1.3) into (1.1), we find that

$$P\omega(x^\circ) = 0 \tag{1.4}$$

This relation is one of the necessary synchronism conditions. After the change of variables suggested in [8]

$$\psi = \frac{1}{\Delta} P\beta, \quad \varphi_k = \frac{1}{\Delta}\beta_{m+k}, \quad k = 1, 2, \ldots, n, \quad n = r - m \tag{1.5}$$

Equation (1.1) is written as

$$\varphi^{\cdot} = a(x) + \varepsilon\Phi(\varphi, \psi, x, \varepsilon), \qquad \psi^{\cdot} = b(x) + \varepsilon\Psi(\varphi, \psi, x, \varepsilon) \qquad x^{\cdot} = \varepsilon X(\varphi, \psi, x) + \varepsilon^2 Y(\varphi, \psi, x, \varepsilon) \tag{1.6}$$

Here φ is a vector with components $\varphi_1, \varphi_2, \ldots, \varphi_{r-m}$, while the vector-valued functions $a(x)$, $b(x)$ are expressed in terms of $\omega(x)$, so that

$$b(x) = \frac{1}{\Delta} P\omega(x), \qquad a_j(x) = \frac{1}{\Delta}\omega_{m+j}(x), \quad j = 1, 2, \ldots, n$$

All the functions occurring in (1.6), naturally, remain sufficiently smooth and 2π-periodic in the components of vectors φ and ψ. The vector $b(x)$ vanishes for $x = x^\circ$, while the components of vector $a(x^\circ)$ are rationally linearly independent.

In the new variables the integral surface (1.2) can be written as

$$x = x^\circ + \varepsilon h(\varphi, \varepsilon), \qquad \psi = \psi^\circ + \varepsilon g(\varphi, \varepsilon)$$

Therefore, the conditions

$$\sum_{j=1}^{n} \frac{\partial h_i(\varphi_1, \varphi_2, \ldots, \varphi_n, 0)}{\partial \varphi_j} \omega_j(x^\circ) = X_i(\varphi_1, \ldots, \varphi_n, \psi_1^\circ, \ldots, \psi_m^\circ, x_1^\circ, \ldots, x_s^\circ)$$

must be fulfilled for $i = 1, 2, \ldots, s$, from which it follows that the mean of the vector-valued function $X(\varphi, \psi^\circ, x^\circ)$ over all the variables $\varphi_1, \varphi_2, \ldots, \varphi_n$ vanishes, i.e.,

$$\langle X(\varphi, \varphi^\circ, x^\circ)\rangle_\varphi = 0 \tag{1.7}$$

Conditions (1.7), as also conditions (1.4), are necessary synchronism conditions.

2. Analysis of the necessary synchronism conditions. Let

$$X(\beta, x) = X(\varphi, \psi, x) = \Sigma X_{k_1, k_2, \ldots, k_r} \exp\left[j \sum_{i=1}^{r} k_i \beta_i\right] = \tag{2.1}$$
$$\Sigma X_{k_1, k_2, \ldots, k_r} \exp\left[j \sum_{i=1}^{n} q_i \varphi_i + j \sum_{i=1}^{m} r_i \psi_i\right]$$

Allowing for the specific form of the connections between β and φ, ψ, we find that

$$k_\alpha = \begin{cases} \varkappa_\alpha, \ \alpha = 1, 2, \ldots, m, \\ \varkappa_\alpha + q_{\alpha-m}, \ \alpha = m+1, m+2, \ldots, r, \end{cases} \qquad \varkappa_\alpha = \frac{1}{\Delta} \sum_{i=1}^{m} p_{i\alpha} r_i \tag{2.2}$$

When averaging (2.1) over all components of vector φ in expansion (2.1) there can remain only those terms in which $q_1 = \ldots = q_n = 0$ or, equivalently, only those terms in which the integer vector k is a linear combination of the row vectors of matrix P $(k \in L(P_1, P_2, \ldots, P_m))$. Hence it follows, in particular, the fulfillment of the inequalities

$$|X_{\alpha, 0, \ldots, 0}(x^\circ)| < \max_\psi \sum_{0 \neq k \in L(P_1, P_2, \ldots, P_m)} |X_{\alpha, k_1, k_2, \ldots, k_r}(\psi, x^\circ)| \tag{2.3}$$

By the order of synchronism we mean the number p^* defined by the formula

$$p^* = \min_{0 \neq k \in L(P_1, P_2, \ldots, P_m)} \max_{1 \leqslant i \leqslant r} |k_i| \tag{2.4}$$

If

$$\sum_{\alpha=1}^{s} X^2_{\alpha, 0, \ldots, 0}(x^\circ) \neq 0$$

and the trigonometric series (2.1) converge absolutely, the fulfillment of estimate (2.3) is possible for not very large values of the order of multiplicity of the synchronism. In the case when the function $X_\alpha(x^\circ, \beta)$ is differentiable some number τ times with respect to the variables $\beta_1, \beta_2, \ldots, \beta_r$, the coefficients of its Fourier series satisfy the estimates

$$|X_{\alpha, k_1, k_2, \ldots, k_r}(x^\circ)| < \max_\beta |D_\beta^\tau X_\alpha(x^\circ, \beta)| \cdot (\max_{1 \leqslant i \leqslant r} |k_i|)^{-\tau} \tag{2.5}$$

As a consequence, inequalities (2.3) take the form

$$|X_{\alpha, 0, \ldots, 0}(x^\circ)| < \max_\beta |D_\beta^\tau X_\alpha(x^\circ, \beta)| \sum_{0 \neq k \in L(P_1, \ldots, P_m)} (\max_{1 \leqslant i \leqslant r} |k_i|)^{-\tau} \tag{2.6}$$

We generalize all we have said on the necessary synchronism conditions in the following theorem.

Theorem 1. For an m-th degree synchronism to exist in a dynamic system described by differential equations (1.1), it is necessary that for certain constant vectors x° and ψ°: (1) the m linearly-independent integer relations (1.4) be fulfilled; (2) the mean value of the function $X(\varphi, \psi^\circ, x^\circ)$ over the collection of variables $\varphi_1, \varphi_2, \ldots, \varphi_n$ equal zero (relations (1.7)), which, in turn, requires the fulfillment of conditions (2.3) possible, in general, only for not very large synchronism orders.

3. Sufficient conditions for existence of synchronism. Let the necessary synchronism conditions (1.4) and (1.7) have been fulfilled. We can treat them as $m+s$ equations in the $m+s$ components of the constant vectors x° and ψ°. To obtain the sufficient existence conditions for synchronism we transform Eqs. (1.6) to equations with constant frequencies of the form

$$\begin{aligned} \varphi^\cdot &= \omega + \varepsilon\Phi(\varphi, v, \varepsilon) \\ v^\cdot &= \varepsilon F(\varphi) + A(\varepsilon)v + \varepsilon L(\varphi)v + \varepsilon V(\varphi, v, \varepsilon) \end{aligned} \tag{3.1}$$

Here all functions are continuous in ε, twice continuously differentiable with respect to the components of vectors φ and v, the components of the constant vector ω are rationally incommensurable, the vector-valued function $V(\varphi, 0, 0)$ and the mean values of the vector-valued function $F(\varphi)$ and of the matrix $L(\varphi)$ over the collection of components of vector φ equal zero, and the estimate

$$\|\exp(A(\varepsilon)\tau)\| < 1 - a\varepsilon\tau, \qquad \|A(\varepsilon)\| < \sqrt{\varepsilon}M \tag{3.2}$$

hold for some $a > 0$ and for sufficiently small $\varepsilon > 0$, $\varepsilon\tau > 0$.

As shown in [10], the system of equations (3.1) admits, under the conditions listed, of a unique stable smooth integral toroidal manifold

$$v = f(\varphi, \varepsilon) \tag{3.3}$$

in some region $\|v\| \leqslant \delta_0$; here the vector-valued function $f(\varphi, \varepsilon)$ is continuous in ε and vanishes for $\varepsilon = 0$.

In system (1.6) we make the change of variables

$$\begin{aligned} x &= x^\circ + \varepsilon x^* + \varepsilon A(\varphi) + \varepsilon z + \varepsilon\sqrt{\varepsilon}C(\varphi)y \\ \psi &= \psi^\circ + \sqrt{\varepsilon}y + \varepsilon\psi^* + \varepsilon B(\varphi) \end{aligned} \tag{3.4}$$

where z and y are the new variables, the vector-valued functions $A(\varphi)$ and $B(\varphi)$, the matrix $C(\varphi)$, the constant vectors x^* and ψ^* are to be defined. In the new variables system (1.6) takes the form

$$\begin{aligned} \varphi^\cdot &= \omega + \varepsilon\Big\{\frac{\partial a(x^\circ)}{\partial x}[x^* + A(\varphi) + z] + \langle\Phi(\varphi, \psi^\circ, x^\circ)\rangle_\varphi + \\ &\quad \Phi^*(\varphi, \psi^\circ, x^\circ)\Big\} + \cdots \\ z^\cdot &= -\frac{\partial A(\varphi)}{\partial\varphi}\Big\{\omega + \varepsilon\Big[\frac{\partial a(x^\circ)}{\partial x}(x^* + A(\varphi) + z) + \\ &\quad \langle\Phi(\varphi, \psi^\circ, x^\circ)\rangle_\varphi + \Phi^*(\varphi, \psi^\circ, x^\circ)\Big]\Big\} - \end{aligned} \tag{3.5}$$

$$\sqrt{\varepsilon}\,\frac{d}{dt}\{C(\varphi)y\}+\langle X(\varphi,\psi^\circ,x^\circ)\rangle_\varphi+X^*(\varphi,\psi^\circ,x^\circ)+$$
$$\varepsilon\langle Y(\varphi,\psi^\circ,x^\circ)\rangle_\varphi+\varepsilon Y^*(\varphi,\psi^\circ,x^\circ)+$$
$$\varepsilon\frac{\partial}{\partial x}[\langle X\rangle_\varphi+X^*][x^*+A(\varphi)+z]+\frac{\partial}{\partial\psi}[\langle X\rangle_\varphi+X^*]\times$$
$$[\sqrt{\varepsilon}y+\varepsilon\psi^*+\varepsilon B(\varphi)]+\ldots$$
$$y^\cdot=-\sqrt{\varepsilon}\,\frac{\partial B(\varphi)}{\partial\varphi}\omega+\sqrt{\varepsilon}\,\frac{\partial b(x^\circ)}{\partial x}[x^*+A(\varphi)+z]+$$
$$\sqrt{\varepsilon}\,\langle\Psi(\varphi,\psi^\circ,x^\circ)\rangle_\varphi+\sqrt{\varepsilon}\,\Psi^*(\varphi,\psi^\circ,x^\circ)+$$
$$\varepsilon\frac{\partial b(x^\circ)}{\partial x}C(\varphi)y+\varepsilon\frac{\partial}{\partial\psi}[\langle\Psi\rangle_\varphi+\Psi^*]y+\ \ldots$$

We simplify system (3.5) by choosing the vector-valued functions $A(\varphi)$ and $B(\varphi)$, as well as the matrix $C(\varphi)$, so as to satisfy the following relations:

$$\frac{\partial A(\varphi)}{\partial\varphi}\omega=X^*(\varphi,\psi^\circ,x^\circ),\qquad\frac{\partial C(\varphi)}{\partial\varphi}\omega=\frac{\partial X^*(\varphi,\psi^\circ,x^\circ)}{\partial\psi}\tag{3.6}$$
$$\frac{\partial B(\varphi)}{\partial\varphi}\omega=\frac{\partial b(x^\circ)}{\partial x}A(\varphi)+\Psi^*(\varphi,\psi^\circ,x^\circ)$$

Here ω denotes the constant vector $a(x^\circ)$, the mean values of the vector-valued functions $X^*(\varphi,\psi^\circ,x^\circ)$ and $\Psi^*(\varphi,\psi^\circ,x^\circ)$ over the collection of variables $\varphi_1, \varphi_2, \ldots, \varphi_n$ equal zero. We assume that the components of the vector $\omega=a(x^\circ)$ satisfy the conditions of strong incommensurability

$$|k_1\omega_1+\ldots+k_n\omega_n|>K(|k_1|+\ldots+|k_n|)^{-p},\ p>0\tag{3.7}$$

The vector-valued functions $A(\varphi)$, $B(\varphi)$ and the matrix $C(\varphi)$ are at least twice continuously differentiable solutions of the system of equations (3.6) if the vector-valued function $X^*(\varphi,\psi^\circ,x^\circ)$ is $(2n+2p+4)$ times continuously differentiable, while the vector-valued function $\Psi^*(\varphi,\psi^\circ,x^\circ)$ and the matrix $\partial X(\varphi,\psi^\circ,x^\circ)/\partial\psi$ are $(n+p+3)$ times continuously differentiable [6].

We choose the vectors x^* and ψ^* as the solution of the system of equations

$$\frac{\partial b(x^\circ)}{\partial x}x^*+\langle\Psi(\varphi,\psi^\circ,x^\circ)\rangle_\varphi=0\tag{3.8}$$
$$\frac{\partial}{\partial x}\langle X(\varphi,\psi^\circ,x^\circ)\rangle_\varphi x^*+\frac{\partial}{\partial\psi}\langle X(\varphi,\psi^\circ,x^\circ)\rangle_\varphi\psi^*+\langle Y(\varphi,\psi^\circ,x^\circ)\rangle_\varphi+\langle Z\rangle_\varphi=0$$

Here $\langle Z(\varphi,\psi^\circ,x^\circ)\rangle_\varphi$ is the mean value of the vector-valued function

$$Z(\varphi,\psi^\circ,x^\circ)=\frac{\partial X^*(\varphi,\psi^\circ,x^\circ)}{\partial x}A(\varphi)+\frac{\partial X^*(\varphi,\psi^\circ,x^\circ)}{\partial\psi}B(\varphi)-$$
$$\frac{\partial A(\varphi)}{\partial\varphi}\left[\frac{\partial a(x^\circ)}{\partial x}A(\varphi)+\Phi^*(\varphi,\psi^\circ,x^\circ)\right]$$

over the collection of variables $\varphi_1, \varphi_2, \ldots, \varphi_n$. It was shown in [11] that under rather general assumptions the system of nonlinear equations (3.8) admits a certain solution (x^*,ψ^*). Thus, system (3.5) takes the following form:

$$\varphi^{\boldsymbol{\cdot}} = \omega + \varepsilon\Phi(\varphi, z, y, \sqrt{\varepsilon}), \quad z^{\boldsymbol{\cdot}} = \varepsilon F_{11}^{*}(\varphi) + \varepsilon\frac{\partial}{\partial x}\langle X(\varphi, \psi^\circ, x^\circ)\rangle_\varphi z + \sqrt{\varepsilon}\frac{\partial}{\partial\psi}\langle X(\varphi, \psi^\circ, x^\circ)\rangle_\varphi y + \varepsilon F_{12}^{*}(\varphi)\, y + \varepsilon Z^*(\varphi, z, y, \sqrt{\varepsilon}) \tag{3.9}$$

$$y^{\boldsymbol{\cdot}} = \sqrt{\varepsilon}\frac{\partial b(x^\circ)}{\partial x} z + \varepsilon\frac{\partial}{\partial\psi}\langle \Psi(\varphi, \psi^\circ, x^\circ)\rangle_\varphi y + \varepsilon F_{21}^{*}(\varphi)\, y + \varepsilon Y^*(\varphi, z, y, \sqrt{\varepsilon})$$

Here the mean values of the vector-valued function $F_{11}^*(\varphi)$, and of the matrices $F_{12}^*(\varphi)$ and $F_{21}^*(\varphi)$ over the collection of variables $\varphi_1, \varphi_2, \ldots, \varphi_n$ equal zero, while the vector-valued functions Z^* and Y^* satisfy the estimate

$$\| Z^*(\varphi, z, y, \sqrt{\varepsilon})\| + \| Y^*(\varphi, z, y, \sqrt{\varepsilon})\| < M(\sqrt{\varepsilon} + \| y \|^2) \tag{3.10}$$

We now introduce the following notation. If the order s of the square matrix $(\partial/\partial x)\times\langle X\rangle_\varphi$ is not less than the order m of the square matrix $(\partial/\partial\psi)\langle\Psi\rangle_\varphi$, then

$$A = \frac{\partial}{\partial x}\langle X\rangle, \quad B = \frac{\partial}{\partial\psi}\langle X\rangle, \quad C = \frac{\partial b}{\partial x}, \quad E = \frac{\partial}{\partial\psi}\langle\Psi\rangle$$

Conversely, if $s < m$, then

$$A = \frac{\partial}{\partial\psi}\langle\Psi\rangle, \quad B = \frac{\partial b}{\partial x}, \quad C = \frac{\partial}{\partial\psi}\langle X\rangle, \quad E = \frac{\partial}{\partial x}\langle X\rangle$$

Thus, by changing if necessary the places of the variables z and y in system (3.9), it is sufficient to investigate the asymptotic nature of the eigenvectors and eigenvalues of the matrix

$$H(\mu) = \begin{Vmatrix} \mu A & B \\ C & \mu E \end{Vmatrix} \qquad (\mu = +\sqrt{\varepsilon})$$

for sufficiently small $0 < \mu < \mu_0$. Here the square matrix A is of order $n = \max(s, m)$, while the square matrix E is of order $r = \min(s, m)$. For the eigenvalues of matrix $H(\mu)$ to have negative real parts, it is necessary that all the eigenvalues of the matrix CB be real and negative [8].

We suppose the fulfillment of the following assumptions [8]:

1) The eigenvalues of matrix CB are real, negative, and distinct. The corresponding numbers

$$l_i = \frac{1}{2\operatorname{Sp}\Omega^*(\lambda_i)} \sum_{\alpha,\beta}\left(e_{\alpha\beta} + \frac{1}{\lambda_i}\sum_{\nu,\delta} c_{\alpha\nu}a_{\nu\delta}b_{\delta\beta}\right)\Omega_{\alpha\beta}^{*}(\lambda_i) \tag{3.11}$$

where the matrix $\Omega^*(\lambda_i)$ is composed of the cofactors of the corresponding elements of the matrix $\Omega(\lambda_i) = CB - \lambda_i I$, are negative.

2) If $n > r$, the equation

$$\Psi(\rho) = \begin{vmatrix} A - I\rho & B \\ C & 0 \end{vmatrix} = 0 \tag{3.12}$$

has $n - r$ distinct roots $\rho_1, \rho_2, \ldots, \rho_{n-r}$, lying to the left of the imaginary axis.

When these assumptions are fulfilled, all the eigenvalues of matrix $H(\mu)$ are distinct and have negative real parts; the matrix composed of the eigenvectors of $H(\mu)$ is nonsingular for $\mu = 0$ [8]. This signifies that the system of equations (3.9) reduces, by a nonsingular transformation of variables, to the form (3.1) with the fulfillment of estimates (3.2). The following theorem holds.

Theorem 2. Assume that:

1) The right hand sides of the system of differential equations (1.1) satisfy the previously-stated conditions of periodicity and smoothness.

2) The necessary synchronism conditions formulated in Theorem 1 are fulfilled for some vectors x° and ψ°.

3) The components of vector $\omega\ (x^\circ)$ satisfy the strong incommensurability conditions (3.7).

4) The solution x^*, ψ^* of system (3.8) exists.

5) The assumptions ensuring the special asymptotic behavior of the eigenvalues and eigenvectors of matrix $H\ (\mu)$ are fulfilled.

Under these assumptions the system of equations (1.1) admits, for sufficiently small ε, of an n-dimensional smooth integral stable toroidal manifold of the form

$$x = x^\circ + \varepsilon x^* + \varepsilon A\ (\varphi) + \varepsilon f_1\ (\varphi, \varepsilon) \qquad (3.13)$$
$$\psi = \psi^\circ + \sqrt{\varepsilon} f_2\ (\varphi, \varepsilon) + \varepsilon\psi^* + \varepsilon B\ (\varphi)$$

unique in the region

$$\| x - x^\circ - \varepsilon x^* - \varepsilon A\ (\varphi) \| \leqslant \varepsilon\delta_0, \quad \| \psi - \psi^\circ - \varepsilon\psi^* - \varepsilon B\ (\varphi) \| \leqslant \sqrt{\varepsilon}\ \delta_0 \qquad (3.14)$$

where δ_0 is some fixed number. The functions $f_1\ (\varphi, \varepsilon)$ and $f_2\ (\varphi, \varepsilon)$ are continuous in ε and tend to zero together with ε.

We note that if the eigenvalues of matrix CB are distinct and if nonzero numbers l_i, defined by formula (3.11), correspond to the real and negative eigenvalues and if Eq. (3.12) has $n - r$ distinct roots lying on both sides of the imaginary axis, then the toroidal integral manifold (3.13) exists also, but is a saddle manifold (*).

4. Synchronism in a quasi-Hamiltonian system. As an application of necessary and sufficient existence conditions for synchronism we consider the synchronization problem in a system of objects described by equations of the form ($\varepsilon > 0$ is a small parameter)

$$\varphi_0^{\cdot} = \omega_0, \quad \varphi_i^{\cdot} = \omega_i\,(J_i) - \varepsilon\left[\frac{\partial X_i}{\partial J_i}\,Q_i + \frac{\partial L}{\partial J_i}\right] + \varepsilon^2\,(\ldots) \qquad (4.1)$$
$$J_i^{\cdot} = \varepsilon\left[\frac{\partial X_i}{\partial \varphi_i}\,Q_i + \frac{\partial L}{\partial \varphi_i}\right] + \varepsilon^2\,(\ldots) \qquad (i = 1, 2, \ldots, n)$$

where

$$X_i = X_i\ (\varphi_i, J_i), \qquad Q_i = Q_i\ (\varphi_i, J_i) \qquad (4.2)$$
$$L = L\ (\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n, J_1, J_2, \ldots, J_n)$$

Periodic solutions (nth-degree synchronism of the first order of multiplicity) have been studied in [12].

Suppose that the first of the necessary synchronism conditions (1.4) has been fulfilled for some vector J°. We write down formula (1.5) defining the change of variables in the new notation of the system of equations (4.1)

---

*) Gurtovnik, A.S., Kogan, V.P. and Neimark, Iu.I., Integral toroidal manifolds in nonlinear systems. Third All-Union Conf. Qualitative Theory of Differential Equations (Reports Abstracts). Samarkand, 1973.

$$\psi_s = \frac{1}{\Delta}\varphi_s, \qquad v_i = \frac{1}{\Delta}\sum_{\alpha=0}^{n} p_{i\alpha}\varphi_\alpha \quad (s = 0, 1, \ldots, n-m;\ i = 1, 2, \ldots, m) \qquad (4.3)$$

System (4.1) takes the form

$$\psi_0^{\boldsymbol{\cdot}} = \frac{\omega_0}{\Delta}, \quad \psi_s^{\boldsymbol{\cdot}} = \frac{1}{\Delta}\left\{\omega_s(J_s) - \varepsilon\left[\frac{\partial X_s}{\partial J_s}Q_s + \frac{\partial L}{\partial J_s}\right]\right\} + \varepsilon^2(\ldots) \qquad (4.4)$$

$$v_i^{\boldsymbol{\cdot}} = \frac{1}{\Delta}\left\{p_{i0}\omega_0 + \sum_{\alpha=1}^{n} p_{i\alpha}\left[\omega_\alpha(J_\alpha) - \varepsilon\left(\frac{\partial X_\alpha}{\partial J_\alpha}Q_\alpha + \frac{\partial L}{\partial J_\alpha}\right)\right]\right\} + \varepsilon^2(\ldots)$$

$$J_k^{\boldsymbol{\cdot}} = \varepsilon\left[\frac{\partial X_k}{\partial \varphi_k}Q_k + \frac{\partial L}{\partial \varphi_k}\right] + \varepsilon^2(\ldots) \qquad (s = 1, \ldots, n-m;\ i = 1, \ldots, m;\ k = 1, \ldots, n)$$

We introduce the following notation:

$$b_i(J) = \frac{1}{\Delta}\left[p_{i0}\omega_0 + \sum_{\alpha=1}^{n} p_{i\alpha}\omega_\alpha(J_\alpha)\right], \qquad A_k(J, v) = \left\langle\frac{\partial X_k}{\partial \varphi_k}Q_k + \frac{\partial L}{\partial \varphi_k}\right\rangle \qquad (4.5)$$

$$R_i(J, v) = -\frac{1}{\Delta}\sum_{\alpha=1}^{n} p_{i\alpha}\left\langle\frac{\partial X_\alpha}{\partial J_\alpha}Q_\alpha + \frac{\partial L}{\partial J_\alpha}\right\rangle \qquad (i = 1, \ldots, m;\ k = 1, \ldots, n)$$

where the symbol $\langle\ \rangle$ denotes averaging over the collection of variables $\psi_0, \ldots, \psi_m$. As has been shown, the existence of vectors J° and v° satisfying the system of nonlinear equations

$$p_{i0}\omega_0 + \sum_{\alpha=1}^{n} p_{i\alpha}\omega_\alpha(J_\alpha) = 0, \qquad A_k(J, v) = 0 \qquad (4.6)$$

$$(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$$

is a necessary condition for an mth-degree synchronism. We assume that system (4.6) admits a certain isolated solution J°, v° for which the components of vector ω_0, $\omega_1(J_1^\circ), \ldots, \omega_{n-m}(J_{n-m}^0)$ satisfy the strong incommensurability conditions (3.7). As before we suppose that the assumptions ensuring the special asymptotic behavior of the eigenvalues and eigenvectors of matrix $H(\mu)$, which in the new notation (4.5) has the form

$$H(\mu) = \left\|\begin{matrix} \mu\dfrac{\partial A}{\partial J} & \dfrac{\partial A}{\partial v} \\ \dfrac{\partial b}{\partial J} & \mu\dfrac{\partial R}{\partial v}\end{matrix}\right\| \qquad (4.7)$$

are fulfilled. By virtue of Theorem 2 an mth-order synchronism obtains in the system of equations (4.1); in other words, system (4.1) admits a stable integral manifold of the form

$$J = J^0 + f(\varphi_0, \varphi_1, \ldots, \varphi_{n-m}\ \mu), \qquad v = v^c + g(\varphi_0, \varphi_1, \ldots, \varphi_{n-m};\ \mu) \qquad (4.8)$$

where the vector-valued functions f and g tend to zero with μ.

Let us examine at somewhat greater length the stability condition for integral manifold (4.8), asserting that all the eigenvalues of the matrix $(\partial b / \partial J)(\partial A / \partial v)$ are negative. In view of the fact that each of the functions X_i and Q_i depends only upon the two variables φ_i and J_i, the following relations hold: (4.9)

$$\frac{\partial A_k}{\partial v_s} = \frac{\partial}{\partial v_s}\left\langle\frac{\partial X_k}{\partial \varphi_k}Q_k + \frac{\partial L}{\partial \varphi_k}\right\rangle = \frac{\partial}{\partial v_s}\left\langle\frac{\partial L}{\partial \varphi_k}\right\rangle \qquad (k = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m)$$

Let $\exp[j(k_0\varphi_0 + k_1\varphi_1 + \ldots + k_n\varphi_n)]$ be any harmonic of function $L(\varphi, J)$ which after the change of variables (4.3) takes the form

$$\exp\left[j\sum_{i=0}^{n-m} q_i\psi_i + i\sum_{i=1}^{m} r_i v_i\right]$$

where $q_0, q_1, \ldots, q_{n-m}, r_1, r_2, \ldots, r_m$ are certain integers. When averaging over the collection of variables $\psi_0, \psi_1, \ldots, \psi_{n-m}$, those and only those harmonics remain in which $q_0 = q_1 = \ldots = q_{n-m} = 0$. By virtue of relations (4.3) this signifies that

$$\left\langle\frac{\partial L}{\partial\varphi_i}\right\rangle = \frac{1}{\Delta}\sum_{s=1}^{m} p_{si}\frac{\partial}{\partial v_s}\Lambda \qquad (i = 1, 2, \ldots, n) \tag{4.10}$$

$$\Lambda = \langle L\rangle_\psi = \left(\frac{1}{2\pi}\right)^{n-m}\int_0^{2\pi}\cdots\int_0^{2\pi} L\,d\psi_0\,d\psi_1\ldots d\psi_{n-m}$$

Thus, for the stability of the integral torus (4.8) it is necessary that the eigenvalues of the matrix

$$\frac{\partial b}{\partial J}\frac{\partial A}{\partial v} = \frac{1}{\Delta^2}P\frac{d\omega}{dJ}P^T S, \qquad S = \left\|\frac{\partial^2}{\partial v_i\partial v_j}\Lambda\right\| \tag{4.11}$$

be real and negative. Here P is a rectangular matrix P_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) of rank m.

Let all the quantities $d\omega_i/dJ_i$ be of the same sign, i.e.

$$\operatorname{sign}\frac{d\omega_1}{dJ_1} = \cdots = \operatorname{sign}\frac{d\omega_n}{dJ_n} = \sigma$$

If the matrix $d\omega/dJ$ is positive- (negative-) definite, then the matrix product $P(d\omega/dJ)P^T$ also is a symmetric positive- (negative-) definite matrix under the condition that the rank of the rectangular matrix P is maximal [13]. The eigenvalues of matrix (4.11) are real, and the signs of the smallest $\lambda_{\min}$ and of the largest $\lambda_{\max}$ are the same as the signs of the smallest $\lambda^*_{\min}$ and of the largest $\lambda^*_{\max}$ eigenvalues of the matrix σS [13]. Thus, for the stability of integral torus (4.8) it is necessary that the symmetric matrix $-\sigma S$ be positive definite.

We seek a potential function $D(J^\circ, v)$ [3, 12] of the form

$$D = -\sigma\left[\Lambda(J^\circ, v) + \sum_{i=1}^{m}\lambda_i(J^\circ)v_i\right]$$

From the form of function D it follows that

$$\frac{\partial}{\partial v_s}D = -\sigma\left[\frac{\partial}{\partial v_s}\Lambda + \lambda_s\right], \qquad \frac{\partial^2 D}{\partial v_i\partial v_j} = -\sigma S$$

By virtue of (4.5) and (4.10) the second of relations (4.6) takes the form

$$A_k(J^\circ, v^\circ) = \left\{\left\langle\frac{\partial X_k}{\partial\varphi_k}Q_k\right\rangle + \frac{1}{\Delta}\sum_{s=1}^{m}p_{sk}\frac{\partial\Lambda}{\partial v_k}\right\}_{J=J^\circ,\ v=v^\circ} = 0 \tag{4.12}$$

From relations (4.12) follows the matrix equality

$$\left\{P\left\langle\frac{\partial X}{\partial\varphi}Q\right\rangle + \frac{1}{\Delta}PP^T\frac{\partial\Lambda}{\partial v}\right\}_{J=J^\circ,\ v=v^\circ} = 0$$

The parameters $\lambda_1(J_1^\circ), \ldots, \lambda_m(J_m^\circ)$ are defined as the solution of the system of linear equations

$$\frac{1}{\lambda} PP^T \lambda = P \left\langle \frac{\partial X}{\partial \varphi} Q \right\rangle_{J=J^\circ}$$

For such a choice of parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ the function $D(J^\circ, v)$ satisfies, at the point $v = v^\circ$, the conditions of stationarity and of strict minimum, based on the analysis of second-order terms, if the matrix $-\sigma Q$ is positive definite.

The necessary and sufficient conditions obtained agree in the particular case of total synchronism with the results obtained in [12].

REFERENCES

1. Beletskii, V. V., Resonance rotation of celestial bodies and Cassini's laws. Celest. Mech., Vol. 6, № 3, 1972.
2. Beletskii, V. V. and Khentov, A. A., Magneto-gravitational stabilization of a satellite. Izv. Akad. Nauk SSSR, MTT, № 4, 1973.
3. Blekhman, I. I., Synchronization of Dynamic Systems. Moscow, "Nauka", 1971.
4. Bogoliubov, N. N. and Zubarev, D. N., The asymptotic approximation method for systems with rotating phase and its application to the motion of a charged particle in a magnetic field. Ukrain. Matem. Zh., Vol. 7, № 1, 1955.
5. Bogoliubov, N. N. and Mitropol'skii, Iu. A., Asymptotic Methods in the Theory of Nonlinear Oscillations. Moscow, Fizmatgiz, 1962.
6. Bogoliubov, N. N., Mitropol'skii, Iu. A. and Samoilenko, A. M., The Method of Accelerated Convergence in Nonlinear Mechanics. Kiev, "Naukova Dumka", 1969.
7. Mitropol'skii, Iu. A. and Samoilenko, A. M., On quasiperiodic oscillations in nonlinear systems. Ukrain. Matem. Zh., Vol. 24, № 2, 1972.
8. Volosov, V. M. and Morgunov, B. I., The Averaging Method in the Theory of Nonlinear Oscillatory Systems. Izd. MGU, 1971.
9. Hale, J. K., Oscillations in Nonlinear Systems. New York, McGraw-Hill Book Co., Inc., 1963.
10. Gurtovnik, A. S. and Neimark, Iu. I., Integral manifolds of differential equations with rapidly-rotating phase. Izv. Vuzov, Radiofizika, Vol. 12, № 11, 1969.
11. Gurtovnik, A. S. and Neimark, Iu. I., Integral manifolds of certain nonlinear systems. Izv. Vuzov, Radiofizika, Vol. 15, № 11, 1972.
12. Nagaev, R. F., Synchronization in a system essentially nonlinear objects with a single degree of freedom. PMM Vol. 29, № 2, 1965.
13. Gantmakher, F. R., Matrix Theory. Moscow, "Nauka", 1966.

Translated by N. H. C.